Embedded Systems
2012/13

Basilica di Santa Maria di Collemaggio, 1287, L'Aquila

Lecture 2
Review of Continuous Processes

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Acknowledgement

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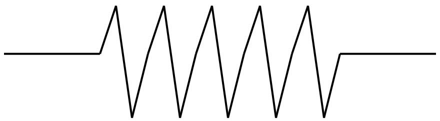
Contents

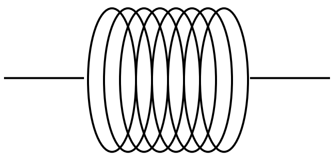
Physical systems

Resistor



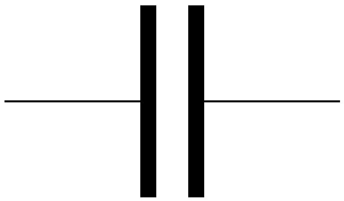
$$V(t) = R \cdot I(t)$$

Inductor



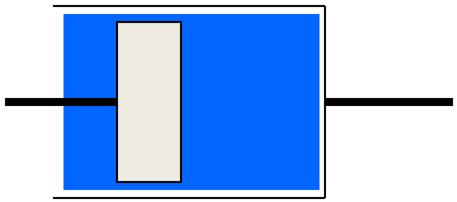
$$V(t) = L\frac{dI}{dt}$$

Capacitor



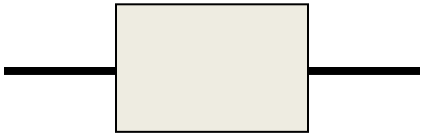
$$I(t) = C\frac{dV}{dt}$$

Damper



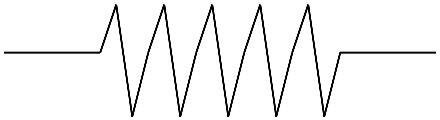
$$F(t) = b \cdot v(t)$$

Mass



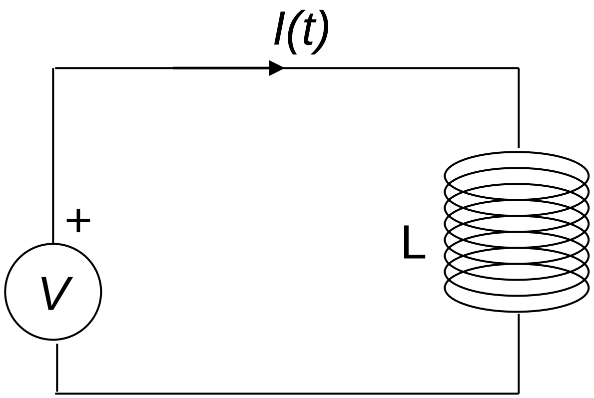
$$F(t) = M\frac{dv}{dt}$$

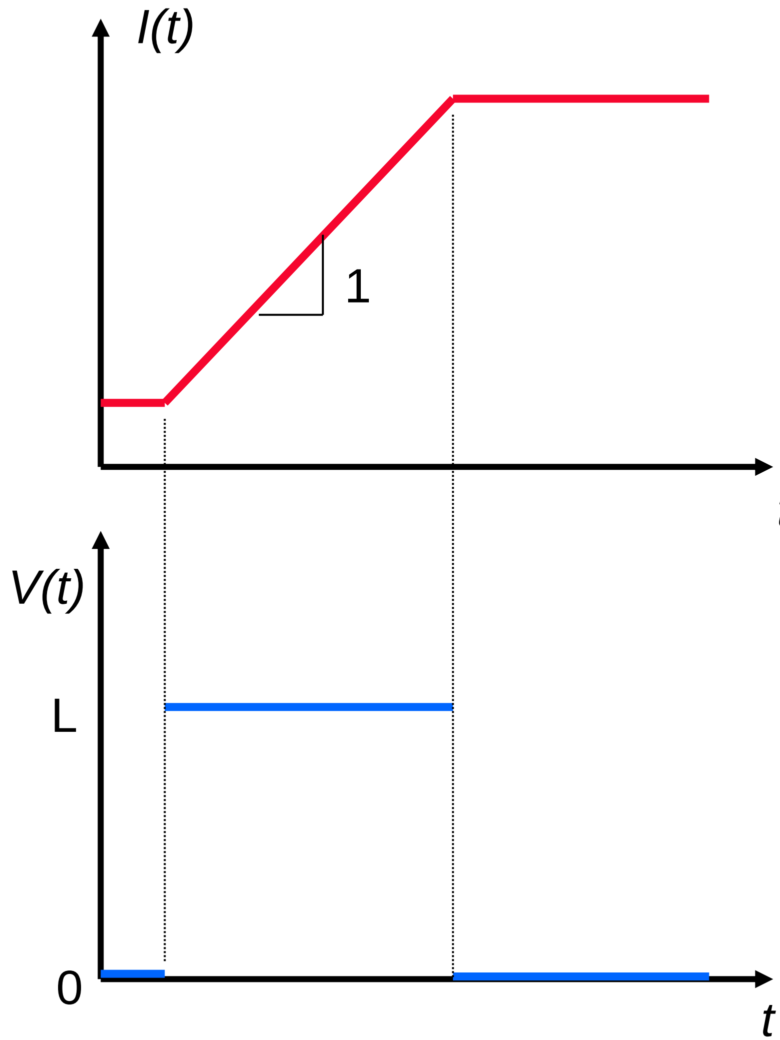
Spring



$$v(t) = \frac{1}{k}\frac{dF}{dt}$$

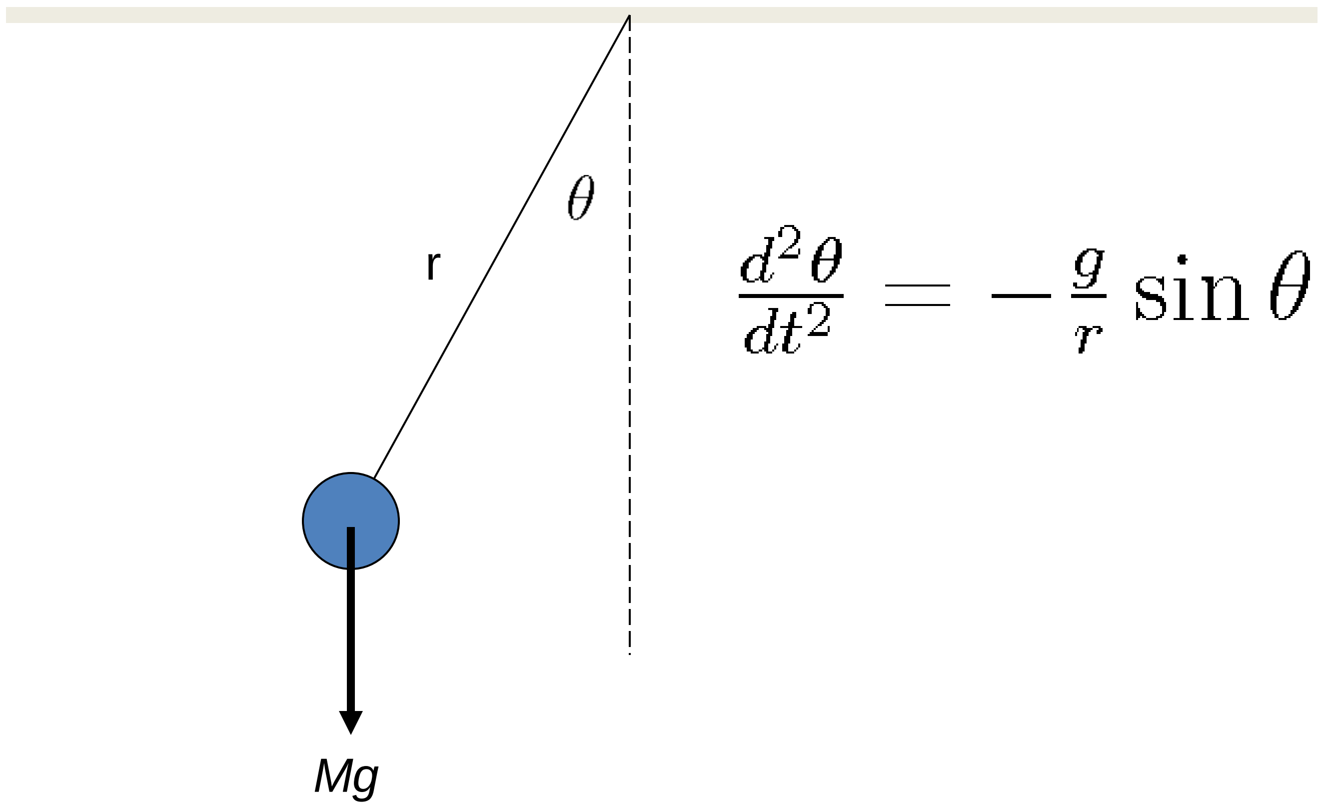
Electric circuits





$$V(t) = L\frac{dI}{dt}$$

A pendulum



$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta(t)$$

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Linear versus nonlinear systems

- Linear systems: if the set of solutions is closed under linear operation, i.e. scaling and addition

$$\left\{\begin{array}{c} V_1(t) = L\frac{dI_1}{dt} \\ V_2(t) = L\frac{dI_2}{dt} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \alpha V_1(t) = L\frac{d(\alpha I_1)}{dt} \\ V_1(t) + V_2(t) = L\frac{d(I_1+I_2)}{dt} \end{array}\right\}$$

- All the examples are linear systems, except for the pendulum

$$\left\{\frac{d^2\theta_1}{dt^2} = -\frac{g}{r}\sin\theta_1(t)\right\} \not\Rightarrow \left\{\frac{d^2\alpha\theta_1}{dt^2} = -\frac{g}{r}\sin\alpha\theta_1(t)\right\}$$

Time-invariant versus time-varying

- Time-invariant: the set of solutions is closed under time shifting

$$\left\{ \frac{d^2\theta_1}{dt^2} = -\frac{g}{r} \sin\theta_1(t) \right\} \Rightarrow \left\{ \frac{d^2\theta_1(t-\Delta)}{dt^2} = -\frac{g}{r} \sin\theta_1(t-\Delta) \right\}$$

- Time varying: the set of solutions is not closed under time shifting

$$\frac{dy}{dt} = tx(t)$$

Autonomous versus non-autonomous

- Autonomous systems: given the past of the signals, the future is fixed

$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta(t)$$

- Non-autonomous systems: there is possibility for input, non-determinism

Contents

- Modeling with differential equations
- Taxonomy of systems
- <u>Solution to linear ODEs</u>
- General solution concept
- Simulation and numerical methods
- State space representation
- Stability

Techniques for autonomous systems

First order linear ODE:

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \gamma x, \\ x(t) &= k \cdot e^{\gamma t}. \end{aligned}}$$

Higher order linear ODEs, denote the differential operator by s,

$$\{\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0\} \Rightarrow \{s^2 + 3s + 2 = 0\}$$

Take the roots of the characteristic polynomial.

$$\boxed{x(t) = k_1 \cdot e^{-2t} + k_2 \cdot e^{-t}}$$

Techniques for non-autonomous systems

Use Laplace transform,

$$\mathcal{L}(x(t)) = X(s) = \int_0^{\infty} x(t)e^{-st}dt.$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) - x(0)$$

Obtain the solution in the frequency domain $X(s)$, and use inverse transform to time domain.

$$\mathcal{L}^{-1}(X(s)) = x(t) = \int_{-\infty}^{+\infty} X(s)e^{st}ds$$

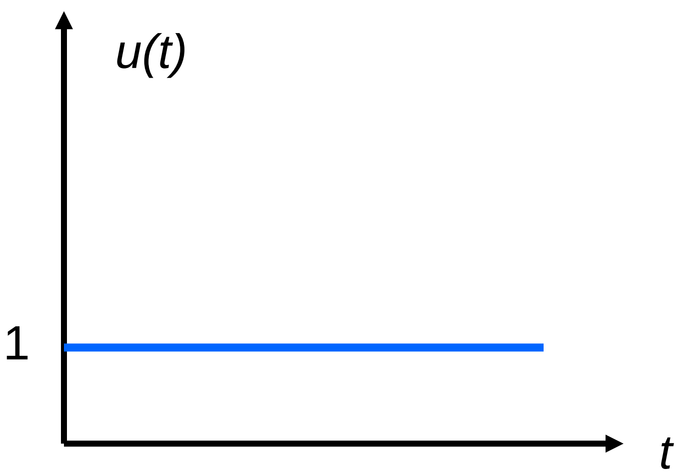
Techniques for non-autonomous systems

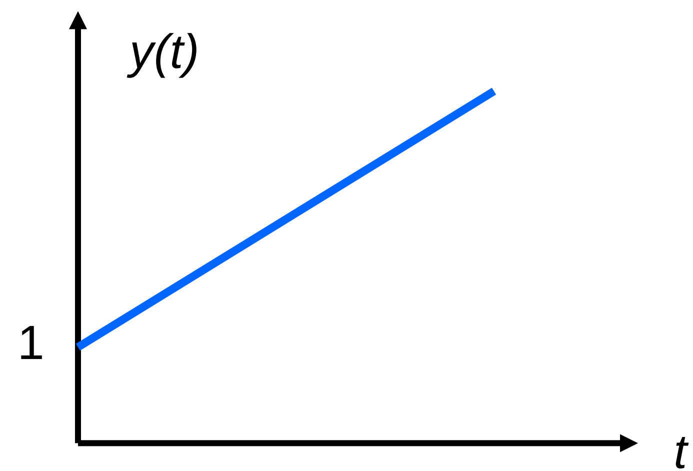
Example:

$$\frac{dy}{dt} = u(t)$$

$$u(t) = \mathbb{1}(t), y(0) = 1.$$

$$sY(s) - 1 = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s^2} + \frac{1}{s}, y(t) \quad = t\mathbb{1}(t) + \mathbb{1}(t).$$





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Solution concepts

- Given a differential equation, $\frac{dx}{dt} = f(x, u)$, and a function $\tilde{x}(t)$. When can we say that $(\tilde{x}(t), \tilde{u}(t))$ is a solution of the differential equation?

- When $\tilde{x}(t)$ is differentiable, then it is straightforward. This is called a strong solution to the equation.

- When $\tilde{x}(t)$ is not differentiable, then $(\tilde{x}(t), \tilde{u}(t)$ is a solution if there exists an x_0 such that

$$\tilde{x}(t) = x_0 + \int_0^t f(\tilde{x}(\tau), \tilde{u}(\tau))\, d\tau$$

This is called a weak solution to the equation.

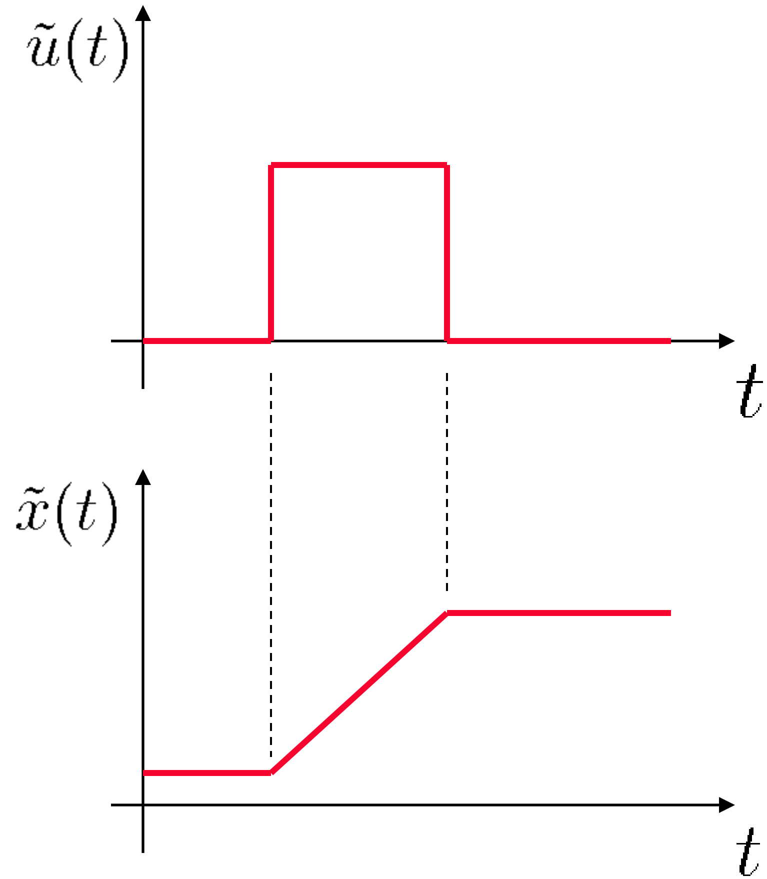
Example of weak solution

Suppose that $\frac{dx}{dt} = u(t)$.

$$\tilde{x}(t) = \begin{cases} 1/4, & t \leq 1 \\ t - 3/4, & 1 < t \leq 2 \\ 5/4, & t > 2 \end{cases},$$

$$\tilde{u}(t) = \begin{cases} 0, & t \leq 1 \\ 1, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases}.$$

is a weak solution since $\tilde{x}(t) = \frac{1}{4} + \int_0^t \tilde{u}(\tau)\, d\tau$.



Existence and uniqueness of solutions

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x over the domain of interest

$f(t, x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities

$f(t, x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r) = \{x \in R^n \mid \|x - x_0\| < r\}$ where $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \;\leq\; L\|x - y\|, \quad L > 0$$

Local existence and uniqueness of solutions

$$\dot{x} = f(t, x)$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

Example: $f(x) = -x^2$ is locally Lipschitz for all x

Global existence and uniqueness of solutions

$$\dot{x} = f(t, x)$$

A function $f(t, x)$ is globally Lipschitz in x if

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y \in R^n$ with the same Lipschitz constant L

Lemma: Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

Example: $f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded

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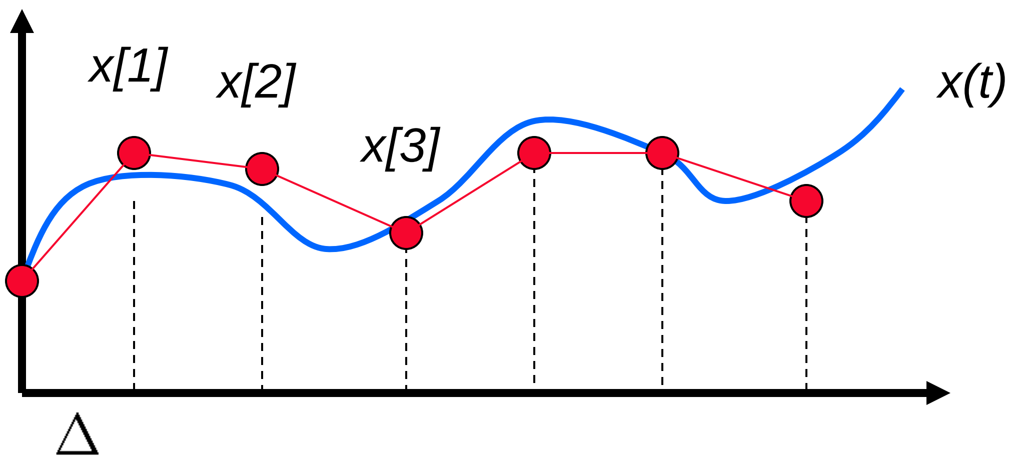
Simulation methods

- Given a differential equation $\frac{dx}{dt} = f(x,t)$.
- To simulate, i.e. numerically compute the solution, we need to discretize.



Forward difference method (Euler) : $\frac{dx}{dt} \approx \frac{x[k+1]-x[k]}{\Delta}$

$$\boxed{x[k+1] = x[k] + \Delta \cdot f(x[k], k\Delta)}$$

Simulation methods

- Backward difference method: $\frac{dx}{dt} \approx \frac{x[k]-x[k-1]}{\Delta}$

$$x[k] = x[k-1] + \Delta \cdot f(x[k], k\Delta)$$

- In each iteration we need to solve an implicit function of $x[k]$. Advantage: the algoritm is more stable.
- Exact discretization is possible for linear time invariant systems.
- There are more sophisticated algoritm, e.g. Runge-Kutta, etc. Most popular algorithms are built in features in most programming/simulation packages, such as MATLAB, MAPLE, etc.

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State space representation

One of the most important representations of linear time-invariant systems.

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

$x(t)$ is called the state of the system, $u(t)$ is the input and $y(t)$ is the output of the system. All variables are vector valued.

A, B, C, D are matrices with appropriate dimensions.

This representation is sometime also called input/state/output representation.

State space representation

- Higher order input/output systems can be cast in state space representation.

$$\boxed{\ddot{y}(t) + 6\dot{y}(t) + 8y(t) = u(t),}$$
$$x_1(t) = y(t), x_2(t) = \dot{y}(t).$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- Thus, we can transform scalar high order ODE to vector first order ODE.

Solution to state space representation

$$\boxed{\begin{aligned}\frac{dx}{dt} &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned}}$$

Solution:

$$\boxed{\begin{aligned}x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau,\\ y(t) &= Ce^{At}x(0) + \int_0^t e^{CA(t-\tau)}Bu(\tau)\,d\tau + Du(t).\end{aligned}}$$

Matrix exponential: $e^A := I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots.$
Easy to compute if A is diagonal.

Alternative: $\boxed{\mathcal{L}(e^{At}) = (sI - A)^{-1}}$

Exact discretization of autonomous systems

- Consider $\dot{x} = Ax(t)$. The solution to this equation is $x(t) = e^{At}x(0)$.
- We sample the system with sampling interval Δ. We have that

$$\boxed{\begin{aligned} x(\Delta) &= e^{A\Delta}x(0), \\ x((k+1)\Delta) &= e^{A\Delta}x(k\Delta), \\ x[k+1] &= e^{A\Delta}x[k]. \end{aligned}}$$